The Flat Plate Problem for a Semiconductor

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A closed-form solution for the potential in a current-free semiconductor surrounding o semi-infinite flat plate, corrying a small potential, is derived. We show that, just as in the case of the classical potential problem, there is a weok (squore-root) singularity in the electric field at the edge of the plate. The solution olso supplies an estimate of the effect of edge curvoture, on edge field for a thick plate, found from equipotential contours. For o plate olong the positive x-oxis the normalized potential u is given by

$$u = e^{y}\,\frac{\operatorname{erfc}(\xi + \eta)}{2} + e^{-y}\left[1 - \frac{\operatorname{erfc}(\xi - \eta)}{2}\right],$$

where erfc *is the complementary error function; ξ, η parabolic coordinates, such that $(\xi + i\eta)^2 = x + iy$; ond x and y are in units of Debye length.*

I. INTRODUCTION

In this paper we calculate the electrostatic potential in a current-free semiconductor surrounding a semi-infinite flat plate carrying a small potential. Our objective is to determine the nature of the singularity in the field at the plate edge. Such information sbould be useful in tbe application of finite difference methods to similar, but more complicated, problems. We find that the field near tbe edge has the same weakly singular behavior as in tbe classical potential problem, being inversely proportional to the square root of tbe distance from the edge.

It turns out, more or less fortuitously, tbat tbe present boundary value problem has a closed form solution, in terms of exponentials and error functions, so tbat it has some intrinsic mathematical interest. Tbe solution was originally obtained by a very tortuous path. The problem was first attacked by the Wiener–Hopf technique. Then it was recognized that the inverse of the Fourier transform of the x-

derivative of the solution had a very simple form. After several changes of integration variable, integration of this derivative led to tho closed form solution.

Here we hypass this involved procedure and derive the solution directly, first factoring a particular solution and then transforming to parabolic coordinates. This after-the-fact derivation is admittedly artificial; in particular, up to now no other nontrivial boundary value problem bas been solved by this "method."

In the sections which follow we first derive the solution, then record various special forms, and finally consider its behavior in the neighborhood of tbe plate edge. The metbod of solution is discussed in Sections II and III; the results are discussed in Sections IV and V.

II. THE BOUNDARY VALUE PROBLEM

The potential $u(x, y)$ due to a semi-infinite flat plate electrode in a current-free semiconductor or a quasi-neutral, stationary plasma satisfies the equation

$$\nabla^2 u = u_{xx} + u_{yy} = u, \tag{1}$$

thc boundary condition

$$u(x, 0) = 1, \quad \text{for} \quad x \geqq 0, \tag{2}$$

for unit potential on the plate, the symmetry condition

$$u_y(x, 0) = 0, \quad \text{for} \quad x < 0, \tag{3}$$

and the limiting condition

$$u(x, y) \to 0, \tag{4}$$

far from the plate. In the ahove the Debye length λ_D has been taken as the unit of length and the plate potential $-\varphi_0(< 0)$ as the unit of potential. The potential $\varphi(X, Y)$ in dimensional form is then given by

$$\varphi(X, Y) = -\varphi_0 u(X/\lambda_D, Y/\lambda_D).$$

Equation (1) only holds for small electrode potential. Specifically, for an n-type semiconductor, we require that

$$q\varphi_0/kT \ll 1,$$

for electronic charge q, Boltzmann constant k, and absolute temperature T. In this case the Debye length is given by

$$\lambda_D = (\epsilon kT/q^2 N_d)^{\frac{1}{2}}$$

for semiconductor dielectric constant ϵ and donor number density N_d.

III. THE SOLUTION

Without preamble we now write down the solution to the above boundary value problem, justifying its form after the fact. Its structure is best displayed in a mixed notation. We find

$$u = e^{y} f(\xi + \eta) + e^{-y}[1 - f(\xi - \eta)], \tag{5}$$

where ξ and η are parabolic coordinates, such that

$$x = \xi^2 - \eta^2, \qquad y = 2\xi\eta,$$

and the function f is given by

$$2f(s) = \operatorname{erfc} s = 1 - \operatorname{erf} s = 2\pi^{-\frac{1}{2}} \int_s^\infty e^{-t^2}\, dt.$$

By using the identities

$$\xi \pm \eta = r^{\frac{1}{2}}\left(\cos\frac{\theta}{2} \pm \sin\frac{\theta}{2}\right),$$

we may also express u in terms of polar coordinates $r = (x^2 + y^2)^{\frac{1}{2}}$, $\theta = \tan^{-1}(y/x)$.

Now let us attempt to unravel this rather involved expression for u. The occurrence of parabolic coordinates is not surprising. They are natural coordinates for flat plate problems with the family of parabolas η = constant degenerating into the positive x-axis for $\eta = 0$ and the orthogonal family ξ = constant giving the negative x-axis for $\xi = 0$. The way in which ξ and η appear in equation (5) and the factors $e^{\pm y}$ are a little more unusual. As we shall see, these two features are tied together.

The exponential factors explicitly display the behavior of the potential far out along the plate, for fixed y and large x, where the potential should approach the one-dimensional potential $e^{-|y|}$. For our purposes they are also essential in simplifying the form of the differential equation in parabolic coordinates. If we transformed equation (1) directly to parabolic coordinates, the resulting equation would be complicated by the presence of the "magnification factor" of the transformation, a function of ξ and η. On the other hand, with an exponential factor removed, the resulting differential equation does not contain the magnification factor. We have a special case of the following general result for equation (1):

Suppose that $u^*(x, y)$ is any particular solution of equation (1). Then $u(x, y) = u^*(x, y)v(x, y)$ also satisfies equation (1) if v satisfies

the equation

$$\nabla^2 v + 2\nabla(\ln u^*)\cdot\nabla v = 0.$$

Now, because the same factor occurs in both the Laplacian and in the product of gradients, this equation has the same form in all orthogonal coordinate systems; that is,

$$v_{\xi\xi} + v_{\eta\eta} + 2[(\ln u^*)_\xi v_\xi + (\ln u^*)_\eta v_\eta] = 0.$$

In general, this identity is of no help, since $\ln u^*$ is a complicated function of ξ and η; but, for $u^* = e^y$, $\ln u^* = y = 2\xi\eta$. Then we have

$$v_{\xi\xi} + v_{\eta\eta} + 4(\eta v_\xi + \xi v_\eta) = 0.$$

Notice the way that ξ and η occur in this equation. If we set

$$v(\xi, \eta) = f(\xi + \eta),$$

the equation reduces to the ordinary differential equation

$$f''(\xi + \eta) + 2(\xi + \eta)f'(\xi + \eta) = 0,$$

an equation satisfied by $f(\xi+\eta) = \text{constant}$ and by $f(\xi+\eta) = \operatorname{erfc}(\xi+\eta)$. A similar result obtains for the multiplier of e^{-y}, except that now we find a function of $\xi - \eta$, $g(\xi - \eta)$, say. To satisfy the boundary condition on the plate, we must have $g(\xi) = 1 - f(\xi)$. We are then left with two arbitrary integration constants in f which are determined from the conditions for fixed y and large positive and negative x. The result is equation (5).

IV. THE POTENTIAL AND FIELD IN POLAR COORDINATES

In polar coordinates equation (6) becomes

$$u = \frac{e^{+r\sin\theta}}{2}\left\{1 - \operatorname{erf}\left[r^{\frac{1}{2}}\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)\right]\right\} + \frac{e^{-r\sin\theta}}{2}\left\{1 + \operatorname{erf}\left[r^{\frac{1}{2}}\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)\right]\right\}. \tag{6}$$

Various special values of u are of particular interest. Along the x-axis we have

$$u(x, 0) = \begin{cases} 1, & \text{for } x \geqq 0 (\theta = 0, r = x), \\ \operatorname{erfc} |x|^{\frac{1}{2}}, & \text{for } x \leqq 0 (\theta = \pi, r = |x|), \end{cases}$$

while on the y-axis ($\theta = \pi/2$, $r = y$)

$$u(0, y) = \frac{e^{y}}{2} \operatorname{erfc} (2y)^{\frac{1}{2}} + \frac{e^{-y}}{2}.$$

Since for large y, $\operatorname{erfc} (2y)^{\frac{1}{2}} \sim (2\pi y)^{-\frac{1}{2}} e^{-2y}$, this vanishes exponentially for large y, as it should.

Near the edge of the plate (that is, for small r)

$$u \approx 1 - 2(r/\pi)^{\frac{1}{2}} \sin \frac{\theta}{2}.$$

The x and y components of electric field are given by

$$u_x = (\pi r)^{-\frac{1}{2}} e^{-r} \sin \frac{\theta}{2}, \tag{7}$$

$$\begin{aligned} u_y = &-(\pi r)^{-\frac{1}{2}} e^{-r} \cos \frac{\theta}{2} \\ &+ \frac{e^{+r \sin \theta}}{2} \left\{ 1 - \operatorname{erf} \left[r^{\frac{1}{2}} \left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right) \right] \right\} \\ &- \frac{e^{-r \sin \theta}}{2} \left\{ 1 + \operatorname{erf} \left[r^{\frac{1}{2}} \left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right) \right] \right\}. \end{aligned} \tag{8}$$

Note the simple form for u_x, a separable solution of equation (1) in polar coordinates. The explicit form for u was in fact originally derived by first calculating u_x and then integrating with respect to x, a rather laborious procedure. According to equations (7) and (8), the field near the edge of the plate is weakly singular; that is, $| \nabla u | \sim (\pi r)^{-\frac{1}{2}}$ for small r, just as in the case of the classical potential problem. Along the x-axis

$$u_x(x, 0) = \begin{cases} 0, & \text{for } x > 0, \\ (\pi \mid x \mid)^{-\frac{1}{2}} e^{-|x|}, & \text{for } x < 0, \end{cases}$$

$$u_y(x, 0) = \begin{cases} -(\pi x)^{-\frac{1}{2}} e^{-x} - \operatorname{erf} (x)^{\frac{1}{2}}, & \text{for } x > 0, \\ 0, & \text{for } x < 0. \end{cases}$$

Note that u_y tends very rapidly along the plate to the one-dimensional value $u_y = -1$. The equipotentials near the plate then rapidly become straight and parallel to the plate. Any one of them can be chosen as an electrode in the form of a thick plate with the electric field large, but not infinite, at its edge, which now has a finite radius of curvature. Thus, at least for this family of plates, we can calculate the maximum field strength as a function of edge curvature.

V. THE EFFECT OF EDGE CURVATURE

Suppose that $x = x(y)$ is the equation of the equipotential $u = u_o$; that is,

$$u[x(y), y] = u_o .$$

Then

$$u_x x' + u_y = 0,$$

and

$$u_x x'' + u_{xx} x'^2 + 2u_{xy} x' + u_{yy} = 0.$$

At the edge $y = 0$

$$x = x(0) = x_o , \qquad x'(0) = 0,$$

where $\operatorname{erfc} | x_o |^{\frac{1}{2}} = u_o$. Thus the radius of curvature of the edge is given by

$$\begin{aligned} R = 1/| x''(0) | &= | u_x(x_o , 0)/u_{yy}(x_o , 0) | \\ &= | u_x(x_o , 0) |/| u(x_o , 0) - u_{xx}(x_o , 0) |, \end{aligned}$$

using the differential equation (1). Now

$$u(x, 0) = \operatorname{erfc} | x |^{\frac{1}{2}},$$

for $x \leqq 0$, so that

$$u_x(x, 0) = e^{-|x|}/(\pi | x |)^{\frac{1}{2}},$$

and

$$u_{xx}(x, 0) = (1 + \tfrac{1}{2} | x |)u_x(x, 0).$$

Thus

$$R = 2 | x_o |/| 1 + 2 | x_o | (1 - 1/w)|, \tag{9}$$

where the normalized gradient

$$w = | \nabla u(x_o , 0) |/u(x_o , 0) = e^{-|x_o|}/(\pi | x_o |)^{\frac{1}{2}} \operatorname{erfc} | x_o |^{\frac{1}{2}}. \tag{10}$$

If equations (9) and (10) are evaluated for various x_o, one obtains the edge field strength due to unit potential applied to a plate with the shape $u = u_o$, as a function of the edge radius of curvature, shown in Fig. 1. As R increases, one might expect that the edge field should approach the field on a circular conductor of radius R. This field, shown as a dashed curve in Fig. 1, is given by $K_1(R)/K_0(R)$, where

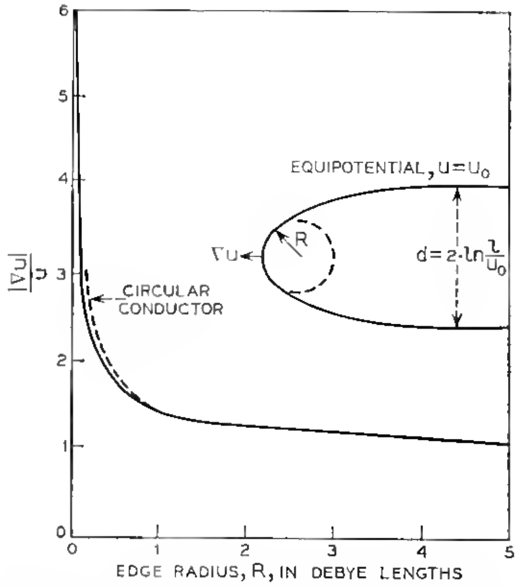

Fig. 1—The edge gradient as a function of edge radius.

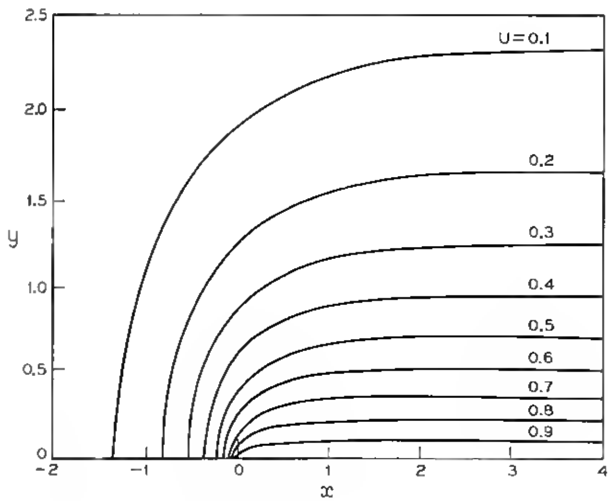

Fig. 2—Equipotentials around a flat plate in a semiconductor.

K_0 and K_1 are the modified Bessel functions of the second kind. Except for very small edge radius, the fields very nearly coincide.

Figure 1 also shows a sketch of a typical equipotential, that is, a thick plate of the family $u = u_o$. The shape shown is typical of moderately small edge radius R. For large R the plate thickness d tends to $2R$, so that thick plates in the family tend to have semicircular noses. On the other hand, thin plates ($u_o \to 1$) have "wedge-shaped" noses, R tending to zero like $(1 - u_o)^2$, while d tends to zero like $1 - u_o$.

Figure 2 shows the exact shapes, for $0.1 \leqq u_o \leqq 1$. Note the sharpness of the equipotential $u = 0.9$, compared with $u = 0.1$, even allowing for the difference in horizontal and vertical scales. For small edge radius then, unless the plate under consideration has a shape similar to the above, the present analysis is not likely to furnish useful estimates of the edge field.

VI. ACKNOWLEDGMENT

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